Solutions 2

2025–26

1. Classify the following equations as parabolic, elliptic or hyperbolic:
   (a) $u_{xx} - u_{xy} + 2u_y + 3u_{yy} - 5u_{yx} + 8u = 0$: Since $(-3)^2 > 1 \cdot 3$, the equation is hyperbolic.
   (b) $9u_{xx} + 6u_{xy} + u_{yy} + u_x = 0$: Since $3^2 = 9 \cdot 1$, the equation is parabolic.
   (c) $u_{xx} - 4u_{xy} + 4u_{yy} = 0$: Since $(-2)^2 = 1 \cdot 4$, the equation is parabolic.

2. Consider the Cauchy problem

$$\begin{cases} u_{tt} = u_{xx}, & x \in \mathbb{R},\ t > 0, \\ u(x,0) = f(x), & u_t(x,0) = g(x). \end{cases}$$

   (a) Find the domain of dependence of u at $(x,t) = (2,1)$.
   (b) Let $f(x) = 0$ outside the interval $[-1,2]$ and $g(x) = 0$ outside the interval $[1,6]$. Find the set E of points (x,t) such that $u(x,t)$ must be zero for $(x,t) \in E$.

   Solution.

   (a) The domain of dependence is $[x - ct, x + ct] = [x - t, x + t] = [2 - 1, 2 + 1] = [1, 3]$.
   (b) Outside the sector for $t > 0$ between lines $x + t = -1$ and $x - t = 6$, i.e. in $\{(x,t) : t > 0, x < -1 - t \text{ or } x > t + 6\}$.

3. Find the solution $u(x,t)$ of the one-dimensional wave equation on an infinite string

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, & x \in \mathbb{R},\ t > 0, \\ u(x,0) = f(x), & u_t(x,0) = g(x). \end{cases}$$

   with

   (a) $f(x) = x$ and $g(x) = \cos(x)$.
   (b) $f(x) = \ln(x^2 + 6)$ and $g(x) = 3x^3$.
   (c) $f(x) = \sin(x^3)$ and $g(x) = \frac{x^2}{x^2+4x+8}$.

   Solution. All of (a), (b) and (c) are solved using d'Alembert's formula

$$u(x,y) = \frac{1}{2}\{f(x+ct) + f(x-ct)\} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\lambda)\mathrm{d}\lambda.$$

   (a) $u(x,y) = \frac{1}{2}\{(x+ct) + (x-ct)\} + \frac{1}{2c}\int_{x-ct}^{x+ct} \cos(\lambda)\mathrm{d}\lambda = x + \frac{1}{2c}(\sin(x+ct) - \sin(x-ct))$.

(b)

$$u(x,y) = \frac{1}{2}\{\ln((x+ct)^2+6) + \ln((x-ct)^2+6)\} + \frac{1}{2c}\int_{x-ct}^{x+ct} 3\lambda^3 \mathrm{d}\lambda.$$
$$= \frac{1}{2}\{\ln((x+ct)^2+6) + \ln((x-ct)^2+6)\} + 3tx(c^2t^2+x^2)$$

(c) The integral is evaluated by conducting polynomial division, or equivalently by noting that $\frac{\lambda^2}{\lambda^2+4\lambda+8} = 1 - \frac{4\lambda+8}{\lambda^2+4\lambda+8}$.

$$u(x,t) = \frac{1}{2}\{\sin((x+ct)^3) + \sin((x-ct)^3)\} + \frac{1}{2c}\int_{x-ct}^{x+ct} \frac{\lambda^2}{\lambda^2+4\lambda+8}\mathrm{d}\lambda$$
$$= \frac{1}{2}\{\sin((x+ct)^3) + \sin((x-ct)^3)\} + t$$
$$+ \frac{1}{c}\left(\ln|(x-ct)^2 + 4(x-ct) + 8| - \ln|(x+ct)^2 + 4(x+ct) + 8|\right).$$

4. Using the method of characteristics, solve the equations

   (a) $2u_x + (\cos x)u_y = 0$, $u(0,y) = e^{-y}$,

   (b) $u_x + 2u_y + (2x-y)u = 2x^2 + 3xy - 2y^2$, $u(x,0) = x$ (harder!).

Solution. (a) We can rewrite this PDE as $(2, \cos x)\cdot(u_x, u_y) = 0$. That is, the directional derivative in the direction $(2, \cos x)$ is zero, i.e. the solution is constant along characteristic curves defined by the ODE

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\cos x}{2}.$$

Therefore the characteristic curves are of the form $y = \frac{1}{2}\sin x + c$, and so solutions to the PDE are of the form $u(x,y) = f(c) = f(y - \frac{1}{2}\sin x)$. The boundary condition implies that $f(z) = \exp(-z)$, so the required solution is $u(x,y) = \exp(-y + \frac{1}{2}\sin x)$.

(b) Consider the curves defined by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 1, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = 2,$$

with conditions $x(0) = s$, $y(0) = 0$. That is,

$$x = t + s, \qquad y = 2t.$$

Along these curves, the PDE reduces to the ODE

$$\frac{\mathrm{d}u}{\mathrm{d}t} + 2su = 2s(s+5t).$$

(Here we have rewritten terms in x and y in terms of t and s.) Multiply by an integrating factor of $\exp(2st)$ to obtain

$$e^{2st}\frac{\mathrm{d}u}{\mathrm{d}t} + 2se^{2st}u = 2s(s+5t)e^{2st} \Leftrightarrow \frac{\mathrm{d}}{\mathrm{d}t}\left\{e^{2st}u\right\} = 2s(s+5t)e^{2st} \Rightarrow e^{2st}u = \frac{(2s^2+10st-5)e^{2st}}{2s} + c(s),$$

(we have used integration by parts) so $u = \frac{2s^2+10st-5}{2s} + c(s)e^{-2st}$. Converting back to original variables x and y gives

$$u(x,y) = x + 2y - \frac{5}{2x-y} + c(x - \frac{y}{2}) \exp\left(-y\left(x - \frac{y}{2}\right)\right).$$

Finally, applying the boundary condition yields that $c(z) = 5/(2z)$, and so

$$u(x,y) = x+2y-\frac{5}{2x-y}+\frac{5}{2x-y}\exp\left(-y\left(x-\frac{y}{2}\right)\right) = x+2y+\frac{5}{2x-y}\left(\exp\left(-y\left(x-\frac{y}{2}\right)\right)-1\right).$$