Assignment 3

2025–26

Starred questions due in by end of Friday 21st November 2025.

1. * Using the reflection method and the d'Alembert formula, solve the problem for the wave equation for a half-infinite string

$$\begin{cases} u_{tt} - 9u_{xx} = 0, \quad x > 0,\ t > 0, \\ u(x,0) = x^2 e^{-x}, \quad u_t(x,0) = xe^{-x}, \\ u(0,t) = 0. \end{cases}$$

2. * Find the solution of the inhomogenous wave equation

$$\begin{cases} u_{tt} - c^2 u_{xx} = x \cos t, & x \in \mathbb{R}, t > 0, \\ u(x,0) = 0, & x \in \mathbb{R} \\ u_t(x,0) = 0, & x \in \mathbb{R}. \end{cases}$$

3. Let $l > 0$ and $m, n \in \mathbb{N}$. Prove that

$$\int_0^l \sin \frac{m\pi x}{l} \sin \frac{n\pi x}{l} \mathrm{d}x = \begin{cases} \frac{l}{2}, & n = m, \\ 0, & n \neq m. \end{cases}$$

   Remark: This was the result we used (then without proof) for the Fourier series part at the end of the finite-length string solution.

4. Using the method of separation of variables, solve the initial boundary value problem

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, \quad x \in [0, \pi],\ t > 0, \\ u(x,0) = \sin 3x, \quad u_t(x,0) = 0, \text{ initial conditions.} \\ u(0,t) = 0, \quad u(\pi,t) = 0,\ t > 0 \text{ boundary conditions.} \end{cases}$$

   (You may use formulae derived in the lectures if you wish to prevent your solutions becoming too long, but make sure you understand their derivation.)

5. * Using the method of separation of variables, find the general solution to the initial boundary value problem for the one dimensional heat equation with endpoints held at zero temperature.

$$\begin{cases} u_t - c^2 u_{xx} = 0, \quad x \in [0, l],\ t > 0, \\ u(x,0) = f(x), \\ u(0,t) = 0, \quad u(l,t) = 0,\ t > 0. \end{cases}$$

   Remark: while the procedure is simliar to that we've seen in the lectures for the wave equation, note that there is only ONE partial derivative w.r.t. t (not two). When formulating the problem for the t variable, bear this in mind.

6. Using the method of separation of variables, find the particular solution to the following Laplace equation initial boundary value problem:

$$\begin{cases} u_{xx} + u_{yy} = 0, & x \in [0,1],\ y \in [0,1]; \\ u(0,y) = u(1,y) = 0, & y \in [0,1]; \\ u(x,1) = 0, & x \in [0,1]; \\ u(x,0) = 4\sin(5\pi x), & x \in [0,1]. \end{cases}$$

Hint: while not essential, you may find that the working is simplified by introducing cosh *and* sinh *functions at some point in your solution.*