Assignment 0: ODEs Refresher

2025–26

Assignment not for handing in

Why?

The method of solution for many PDEs (Partial Differential Equations) that we'll see in this module can be broadly summarised as follows: we'll do something to the PDE that reduces it to an ODE (ordinary differential equation) and then solve the resulting ODE. This is great, but only if we remember how to solve some ODEs. This assignment should refresh your memory on some simple types which will come up again and again. Time invested here will help you significantly later.

Some practice ODEs questions

1. **First order separable ODEs.**

   Find the general solution of each ODE by separation of variables:

   (a) $\frac{\mathrm{d}y}{\mathrm{d}x} = xy$.

   (b) $\frac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{2x}{y}$.

   (c) $\frac{\mathrm{d}y}{\mathrm{d}x} = e^{x-y}$.

2. **First order linear ODEs.**

   Solve each using an integrating factor:

   (a) $\frac{\mathrm{d}y}{\mathrm{d}x} + y = x$.

   (b) $\frac{\mathrm{d}y}{\mathrm{d}x} + \dfrac{2}{x}y = 3 \qquad (x > 0)$.

   (c) $\frac{\mathrm{d}y}{\mathrm{d}x} + 4xy = 8x$.

3. **Second order linear ODEs with constant coefficients.**

   For each of the following homogeneous ODEs, write down and solve the *auxiliary equation* and hence give the *complementary function*. Include one of each root type.

   (a) $y'' - y' - 2y = 0$.

   (b) $y'' - 4y' + 4y = 0$.

   (c) $y'' + 4y' + 13y = 0$.

4. **Which method would you use?**

   For each ODE, state a suitable method *(no need to solve)*:

   (a) $\frac{\mathrm{d}u}{\mathrm{d}x} = x^3 - 2x$.

   (b) $\frac{\mathrm{d}u}{\mathrm{d}x} + \dfrac{1}{x}u = 5x$.

   (c) $u'' + 5u' + 6u = 0$.

   (d) $u' - 2u = e^{2x}$.

   (e) $u' = x^2u^2$.

5. **Initial/boundary value problems.** For each, find the *particular solution* that satisfies the given condition(s).

   (a) $\frac{\mathrm{d}y}{\mathrm{d}x} + y = x, \quad y(0) = 1$.

   (b) $\frac{\mathrm{d}y}{\mathrm{d}x} = xy, \quad y(0) = 2$.

   (c) $y'' - y' - 2y = 0, \quad y(0) = 1, y'(0) = 0$.

Model Solutions

1. **First order separable ODEs.**

   (a) $\frac{\mathrm{d}y}{\mathrm{d}x} = xy$.
   This is separable. Rearrange (putting everything involving y on the left, everything involving x on the right) and integrate both sides to obtain

   $$\int y^{-1}\mathrm{d}y = \int x\mathrm{d}x \Rightarrow \ln|y| = \tfrac{1}{2}x^2 + C,$$

   where C is an arbitrary constant. Exponentiate and absorb sign into the constant:

   $$y = Ae^{x^2/2}, \quad A \in \mathbb{R}.$$

   (b) $\frac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{2x}{y}$.
   This is separable, so

   $$\int y\mathrm{d}y = \int 2x\mathrm{d}x \Rightarrow \tfrac{1}{2}y^2 = x^2 + C \Rightarrow y^2 = 2x^2 + C_1,$$

   where C and C_1 are arbitrary constants. Hence,

   $$y(x) = \pm\sqrt{2x^2 + C_1}.$$

   (c) $\frac{\mathrm{d}y}{\mathrm{d}x} = e^{x-y} = e^x e^{-y}$.
   This is separable. Rearranging and integrating,

   $$\int e^y\mathrm{d}y = \int e^x\mathrm{d}x \Rightarrow e^y = e^x + C,$$

   where C is an arbitrary constant. Therefore

   $$y(x) = \ln\big(e^x + C\big).$$

2. **First order linear ODEs (integrating factor).**

   (a) $\frac{\mathrm{d}y}{\mathrm{d}x} + y = x$.
   Multiply by an integrating factor

   $$e^{\int 1\mathrm{d}x} = e^x$$

   to obtain

   $$e^x\frac{\mathrm{d}y}{\mathrm{d}x} + e^x y = xe^x \Leftrightarrow \frac{\mathrm{d}}{\mathrm{d}x}\{e^x y\} = xe^x.$$

   Note that the equivalence in the equation above is more easily seen in the backward direction by differentiating the expression in curly brackets using the product rule.
   Integrate and rearrange:

   $$e^x y = \int xe^x\mathrm{d}x + C = (x-1)e^x + C \Leftrightarrow y = x - 1 + Ce^{-x}.$$

   Here, C is an arbitrary constant, and the integration is conducted by parts.

(b) $\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{2}{x}y = 3$ $(x > 0)$.
Multiply by an integrating factor

$$e^{\int 2x^{-1}\mathrm{d}x} = e^{2\ln x} = (e^{\ln x})^2 = x^2$$

to obtain

$$x^2\frac{\mathrm{d}y}{\mathrm{d}x} + 2xy = 3x^2 \Leftrightarrow \frac{\mathrm{d}}{\mathrm{d}x}\{x^2 y\} = 3x^2.$$

Integrating gives

$$x^2 y = x^3 + C \Rightarrow y = x + Cx^{-2},$$

where C is an arbitrary constant.

(c) $\frac{\mathrm{d}y}{\mathrm{d}x} + 4xy = 8x$.
Multiply by an integrating factor

$$e^{\int 4x\mathrm{d}x} = e^{2x^2}$$

to obtain

$$e^{2x^2}\frac{\mathrm{d}y}{\mathrm{d}x} + 4xe^{2x^2}y = 8xe^{2x^2} \Leftrightarrow \frac{\mathrm{d}}{\mathrm{d}x}\left\{e^{2x^2}y\right\} = 8xe^{2x^2}.$$

Integrate (note $\frac{\mathrm{d}}{\mathrm{d}x}(e^{2x^2}) = 4xe^{2x^2}$):

$$e^{2x^2}y = \int \mathrm{d}x 8xe^{2x^2}\mathrm{d}x + C = 4e^{2x^2} + C,$$

where C is an arbitrary constant, hence

$$y = 4 + Ce^{-2x^2}.$$

3. **Second order linear ODEs with constant coefficients.**

(a) $y'' - y' - 2y = 0$.
Auxiliary equation: $m^2 - m - 2 = 0 \Rightarrow (m-2)(m+1) = 0$, so $m_1 = 2$, $m_2 = -1$ (real and distinct). The complementary function is

$$y_c(x) = Ae^{2x} + Be^{-x}.$$

(b) $y'' - 4y' + 4y = 0$.
Auxiliary equation: $m^2 - 4m + 4 = 0 \Rightarrow (m-2)^2 = 0$, a repeated root $m = 2$ (multiplicity 2). Then

$$y_c(x) = (A + Bx)e^{2x}.$$

(c) $y'' + 4y' + 13y = 0$.
Auxiliary equation: $m^2 + 4m + 13 = 0$. Roots

$$m_\pm = -2 \pm 3i \quad (\alpha = -2, \beta = 3).$$

Hence

$$y_c(x) = e^{\alpha x}(A\cos(\beta x) + B\sin(\beta x)) = e^{-2x}(A\cos 3x + B\sin 3x).$$

4. **Which method would you use?**

   (a) $\frac{\mathrm{d}u}{\mathrm{d}x} = x^3 - 2x$. *Direct integration w.r.t. x* (of the form $\frac{\mathrm{d}u}{\mathrm{d}x} = f(x)$).

   (b) $\frac{\mathrm{d}u}{\mathrm{d}x} + \dfrac{1}{x}u = 5x$. *Integrating factor* (first order linear).

   (c) $u'' + 5u' + 6u = 0$. *Complementary function* via the *auxiliary equation* (second order, constant coefficients, homogeneous).

   (d) $u' - 2u = e^{2x}$. *Integrating factor* (first order linear).

   (e) $u' = x^2u^2$. *Separation of variables* (of the form $\frac{\mathrm{d}u}{\mathrm{d}x} = f(x)g(u)$).

5. **Initial/boundary value problems.**

   (a) $\frac{\mathrm{d}y}{\mathrm{d}x} + y = x, \quad y(0) = 1.$
   From 2(a), the general solution is $y = x - 1 + Ce^{-x}$. Impose $y(0) = 1$:

   $$1 = 0 - 1 + C \Rightarrow C = 2.$$

   Hence the particular solution is

   $$y(x) = x - 1 + 2e^{-x}.$$

   (b) $\frac{\mathrm{d}y}{\mathrm{d}x} = xy, \quad y(0) = 2.$
   From 1(a), the general solution is $y = Ce^{x^2/2}$. Impose $y(0) = 2$:

   $$2 = Ce^0 \Rightarrow C = 2.$$

   Therefore

   $$y(x) = 2e^{x^2/2}.$$

   (c) $y'' - y' - 2y = 0, \quad y(0) = 1, y'(0) = 0.$
   From 3(a), $y_c(x) = Ae^{2x} + Be^{-x}$. Then

   $$y'(x) = 2Ae^{2x} - Be^{-x}.$$

   Apply the conditions at $x = 0$:

   $$y(0) = A + B = 1, \qquad y'(0) = 2A - B = 0.$$

   Solve: from the second, $B = 2A$. Substitute into the first: $A + 2A = 1 \Rightarrow A = \frac{1}{3}$, hence $B = \frac{2}{3}$. So

   $$y(x) = \tfrac{1}{3}e^{2x} + \tfrac{2}{3}e^{-x}.$$