PRIFYSGOL ABERYSTWYTH UNIVERSITY

ADRAN MATHEMATEG / DEPARTMENT OF MATHEMATICS

ARHOLIADAU SEMESTER 1 / SEMESTER 1 EXAMINATIONS

IONAWR / JANUARY 2025

MA34110 – Partial Differential Equations

The questions on this paper are written in English.

Amser a ganiateir - 2 awr

- Gellir rhoi cynnig ar bob cwestiwn.
- Rhoddir mwy o ystyriaeth i berfformiad yn rhan B wrth bennu marc dosbarth cyntaf.
- Cyfrifianellau Casio FX-83 neu FX-85 YN UNIG a ganiateir.
- Mae modd i fyfyrwyr gyflwyno atebion i'r papur hwn naill ai yn y Gymraeg neu'r Saesneg.

Time allowed - 2 hours

- All questions may be attempted.
- Performance in section B will be given greater consideration in assigning a first class mark.
- Casio FX-83 or FX-85 calculators ONLY may be used.
- Students may submit answers to this paper in either Welsh or English.

Section A

1. A partial differential equation (PDE) can be written in the form

$$Lu = g,$$

where L is an operator, u is the dependent variable and g is a known function of the independent variables.

(a) What can be said about L and g if the PDE is *linear homogeneous*? [2 marks]

(b) For each of the following equations, state the order and classify it as nonlinear, linear inhomogeneous, or linear homogeneous:

(i) $u_x - 3u_x u_y + u_{yy} = \cos y$;

(ii) $u_x + u_y = 2x^2 \cosh y$;

(iii) $(u_x)^2 + (u_y)^2 = 2\sin x$;

(iv) $u_x - 2u_{xxx} + 3xu_x = 1$. [8 marks]

(c) Classify the following PDEs as parabolic, elliptic or hyperbolic:

(i) $4u_{xx} + 2u_{xy} + 6u_{yy} + 2u_x - 3u = 0$;

(ii) $3u_{xx} + 12u_{xy} + 12u_{yy} - 2u_x + 7u_y - u = 0$. [4 marks]

(d) Find the general solution $u = u(x, y)$ of the PDE $u_{xy} = 8xy + 1$. [3 marks]

2. Suppose the temperature, u, of a thin rod satisfies the heat equation $u_t = u_{xx}$ for $(x, t) \in (0, 3) \times (0, 6)$, with the following initial and boundary conditions:

$$\begin{cases} u(x,0) = \frac{1}{3}(x^2 - 2x), & x \in [0,3]; \\ u(0,t) = \sin\left(\frac{\pi}{12}t\right), & t \in [0,6]; \\ u(3,t) = e^{-t/2}, & t \in [0,6]. \end{cases}$$

(a) Specifically for the domain defined above, state the definition of the *parabolic boundary*, Π. [3 marks]

(b) State the maximum and minimum principles for the heat equation. [2 marks]

(c) Without solving the equation and stating clearly any results you use, find the minimum and maximum temperatures of the rod for all $(x, t) \in (0, 3) \times (0, 6)$. [8 marks]

(d) State the values of x and t for which the temperature is minimised. [2 marks]

3. Use the method of characteristics to solve the following boundary value problems:

(a) $\begin{cases} u_x - 6u_y = 0, \\ u(0, y) = y^2; \end{cases}$ (b) $\begin{cases} xu_x + 2yu_y = 0, \\ u(1, y) = \cos y. \end{cases}$

You should explain the steps in your working throughout. [6,8 marks]

4. Consider the following one-dimensional heat equation problem, describing the temperature of a one-dimensional rod of length L with insulated ends, whose initial temperature distribution is given by some known function f:

$$\begin{cases} u_t - c^2 u_{xx} = 0, & (x,t) \in (0,L) \times (0,T); \\ u(0,t) = u(L,t) = 0, & t \in [0,T]; \\ u(x,0) = f(x), & x \in [0,L]. \end{cases} \tag{1}$$

(a) Using the method of separation of variables, find a solution of problem (1) in the form

$$u(x,t) = \sum_{n=1}^{\infty} X_n(x) T_n(t).$$

[11 marks]

(b) Hence, or otherwise, solve problem (1) with $c = 1$, $L = T = \pi$ and $f(x) = 3\sin(8x)$. [4 marks]

5. Consider the Cauchy problem for the homogeneous wave equation on an infinite string:

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, & x \in \mathbb{R},\ t > 0; \\ u(x,0) = f(x), & x \in \mathbb{R}; \\ u_t(x,0) = g(x), & x \in \mathbb{R}. \end{cases} \tag{2}$$

(a) What property of the string does the constant c describe? [1 mark]

(b) Suppose g is the zero function and $f \in C^2(\mathbb{R})$ is such that $f(x)$ is non-zero for all $x \in [0,1]$ and is zero for all x outside this interval. Stating clearly any facts about wave equation solutions you use, give in terms of c both the smallest and largest positive values of t for which $u(5,t)$ is non-zero. [3 marks]

(c) Solve problem (2) for all $x \in \mathbb{R}$, $t > 0$, with $c = 1$, and f, g given by

$$f(x) = \sin(2x), \qquad g(x) = 2x^2.$$

You do not need to simplify trigonometric expressions, but you should leave polynomials in their simplest possible form.

[5 marks]

SECTION B BEGINS ON THE NEXT PAGE

Section B

6. Use the method of characteristics to find the solution of the PDE

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=u,$$

which satisfies the condition $u(x,1)=x^2$. Give your answer in as simple a form as possible. [9 marks]

7. By considering the energy integral

$$E(t)=\frac{1}{2}\int_0^l u^2(x,t)\mathrm{d}x,$$

prove that the following heat equation problem has a unique solution.

$$\begin{cases} u_t=c^2u_{xx}, & (x,t)\in(0,l)\times(0,T);\\ u(x,0)=f(x), & x\in[0,l];\\ u(0,t)=\phi(t), & t\in[0,T];\\ u(l,t)=\psi(t), & t\in[0,T]. \end{cases}$$

You may assume existence without proof. [7 marks]

8. Let u be a solution to the Cauchy problem for the heat equation:

$$\begin{cases} u_t-c^2u_{xx}=0, & x\in\mathbb{R},\ t>0;\\ u(x,0)=f(x), & x\in\mathbb{R}. \end{cases} \tag{3}$$

(a) Clearly stating any results that you use, show that the Fourier transform (taken with respect to x) of u satisfies

$$\frac{\partial\bar{u}}{\partial t}+c^2\xi^2\bar{u}(\xi,t)=0.$$

[4 marks]

(b) Hence find an explicit expression for $\bar{u}(\xi,t)$, the Fourier transform of the solution to problem (3), in terms of $\bar{f}(\xi)$. [5 marks]

9. (a) List three properties that a *well-posed* problem in the sense of Hadamard must possess. [3 marks]

(b) By seeking a separable solution, show that the following problem is *ill-posed* in the sense of Hadamard:

$$\begin{cases} u_{xx}-tu_t+2u=0, & x\in(0,\pi),\ t\in(0,1);\\ u(0,t)=0, & t\in[0,1];\\ u(\pi,t)=0, & t\in[0,1];\\ u(x,0)=0, & x\in[0,\pi]. \end{cases}$$

[13 marks]

10. Given that the problem

$$\begin{cases} u_t - c^2 u_{xx} = 0, & x \in \mathbb{R},\ t > 0; \\ u(x,0) = \phi(x), & x \in \mathbb{R}; \end{cases}$$

has solution given by

$$u(x,t) = \frac{1}{\sqrt{4\pi c^2 t}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4c^2 t} \phi(y) \mathrm{d}y,$$

derive the solution to

$$\begin{cases} v_t - c^2 v_{xx} = 0, & x > 0,\ t > 0; \\ v(x,0) = \psi(x), & x > 0; \\ v(0,t) = 0, & t > 0. \end{cases}$$

You should justify any assertions you make throughout your derivation. [9 marks]

END OF EXAM PAPER