S = seen similar
B = bookwork
U = unseen

MA34110 : Partial Differential Equations – January 2023 paper

SECTION A

Q1 CLASSIFICATION S

a) (i) 3^{rd} order, linear homogeneous. 2

(ii) 2^{nd} order, linear inhomogeneous. 2

(iii) 1^{st} order, non-linear. 2

(iv) 2^{nd} order, non-linear. 2

b) (i) $a_{11} = 1$, $a_{12} = 2$, $a_{22} = 4$, so $a_{11}a_{22} = 1 \times 4 = 2^2 = a_{12}^2$; parabolic. 2

(ii) $a_{11} = 9$, $a_{12} = 6$, $a_{22} = -2$, so $a_{11}a_{22} = -18 < 6^2 = a_{12}^2$; hyperbolic. 2

c) Integrate w.r.t. y : $u_x = \frac{xy^2}{2} + f_1(x)$ (f_1 an arbitrary function)

Integrate w.r.t. x : $u = \frac{x^2y^2}{4} + f(x) + g(y)$ ($f, g : \mathbb{R} \to \mathbb{R}$ arbitrary functions). 3

15 Q1

Q2 METHOD OF CHARACTERISTICS S

a) Rewriting as $(1, 7) \cdot (u_x, u_y) = 0$, we see that the directional derivative of u in the direction $(1, 7)$ is zero. These characteristic lines are defined by $\frac{dy}{dx} = 7 \Rightarrow y = 7x + c$ $(c \in \mathbb{R})$. 4 u is constant along each such line, so $u = f(c) = f(y - 7x)$, where $f : \mathbb{R} \to \mathbb{R}$ is an arbitrary function.

Applying the boundary condition gives $u(0,y) = f(y) = y^2$, so $\boxed{u(x,y) = (y - 7x)^2.}$ 3

b) Rewriting as $(1, \cos x) \cdot (u_x, u_y) = 0$, we see that the directional derivative of u in the direction $(1, \cos x)$ is zero. Curves with such tangents are defined by $\frac{dy}{dx} = \cos x \Rightarrow y = \sin x + c$ $(c \in \mathbb{R})$. 5 u is constant along each such curve, so $u = f(c) = f(y - \sin x)$, where $f : \mathbb{R} \to \mathbb{R}$ is an arbitrary function.

Applying the boundary condition gives $u(\pi, y) = f(y - 0) = f(y) = \sin y$, so $\boxed{u(x,y) = \sin(y - \sin x).}$ 4

16 Q2

Q3. WAVE EQUATION S

By linearity of the wave equation, $u = v + w$, where v and w solve:

$$① \begin{cases} v_{tt} - v_{xx} = 0, & x \in \mathbb{R}, t > 0; \\ v(x,0) = f(x), & x \in \mathbb{R}; \\ v_t(x,0) = g(x), & x \in \mathbb{R}. \end{cases} \qquad ② \begin{cases} w_{tt} - w_{xx} = h(x,t), & x \in \mathbb{R}, t > 0; \\ w(x,0) = 0, & x \in \mathbb{R}; \\ w_t(x,0) = 0, & x \in \mathbb{R}. \end{cases}$$

Problem ① has solution given by d'Alembert's formula, problem ② by Duhamel's. 4

Let us solve problem ①:

$$v(x,t) = \tfrac{1}{2}\{f(x+t) + f(x-t)\} + \tfrac{1}{2}\int_{x-t}^{x+t} g(\lambda)\,d\lambda$$
$$= (x+t)^2 + (x-t)^2 + \tfrac{1}{2} \times 2\left[e^{\lambda/2}\right]_{\lambda = x-t}^{\lambda = x+t}$$
$$= 2(x^2 + t^2) + e^{\frac{1}{2}(x+t)} - e^{\frac{1}{2}(x-t)}.$$

4

Next, solve problem ②:

$$w(x,t) = \tfrac{1}{2}\int_0^t \int_{x-t+\tau}^{x+t-\tau} 12y\tau^2\,dy\,d\tau = \int_0^t \left(\int_{x-t-\tau}^{x+t-\tau} 6y\,dy\right)\tau^2\,d\tau$$
$$= 3\int_0^t \left((x+t-\tau)^2 - (x-t+\tau)^2\right)\tau^2\,d\tau = 3\int_0^t 2x(2t - 2\tau)\tau^2\,d\tau$$
$$= 12\int_0^t x(t-\tau)\tau^2\,d\tau = 12x\left(t\int_0^t \tau^2\,d\tau - \int_0^t \tau^3\,d\tau\right) = 12x\left(\tfrac{t}{3}[\tau^3]_0^t - \tfrac{1}{4}[\tau^4]_0^t\right)$$
$$= 12xt^4\left(\tfrac{1}{3} - \tfrac{1}{4}\right) = xt^4.$$

4

Combining the above, $\boxed{u(x,t) = 2(x^2 + t^2) + e^{\frac{1}{2}(x+t)} - e^{\frac{1}{2}(x-t)} + xt^4.}$ (Other simplifications also accepted) 2

14 Q3

Q4.

MAXIMUM PRINCIPLE FOR HEAT EQUATION

S

a) $\Pi = (\{0\} \times [0,7]) \cup ([0,2] \times \{0\}) \cup (\{2\} \times [0,7])$. 3

b) Let $u = u(x,t)$ be a solution of the heat equation on $[0,2]\times[0,7]$. Then the minimal and maximal values of u occur on the parabolic boundary Π. 2

c) The minimum and maximum principles yield that u's minimum and maximum values lie on Π. We therefore minimise and maximise u on Π:

- $2x - x^2$ over $[0,2]$:

  since $2x - x^2 = x(2-x)$, its minimum value is 0, obtained at $x=0$ and $x=2$, its maximum value is 1, obtained at $x=1$. 3

- $g(t) = \frac{8t}{1+t}$ over $[0,7]$:

  note $g'(t) = \frac{8}{(1+t)^2} > 0 \;\forall t$, so the function is strictly monotone increasing. Thus its minimum value is 0, obtained at $t=0$, and its maximum value is 7, obtained at $t=7$. 3

- (and trivially 0 on the remaing part of Π).

Combining the above, $\boxed{0 \leqslant u(x,t) \leqslant 7 \quad \text{for all } (x,t) \in A.}$ 2

d) $x = 2$, $t = 7$. 2

15 Q4

Q5

SEPARABLE HEAT EQ.

S/B

a) Seek solutions in form $u(x,t) = X(x)T(t)$.
Substitute into heat equation to give $\frac{X''(x)}{X(x)} = \frac{T'(t)}{c^2 T(t)} = -\lambda$, say $(\lambda \in \mathbb{R})$,
Since LHS is a function only of x, RHS only of t. 2

Consider the problem for x: $X'' + \lambda X = 0$, $X(0) = 0$, $X(\ell) = 0$.
For non-trivial solutions, $\lambda > 0$, so let $\lambda = \omega^2$, giving $X(x) = A\cos(\omega x) + B\sin(\omega x)$.
Applying boundary conditions: $X(0) = 0 \Rightarrow A = 0$, $X(\ell) = 0 \Rightarrow \omega_k = \frac{k\pi}{\ell}$, $k \in \mathbb{N}$. Thus $X_k(x) = B_k \sin\left(\frac{k\pi x}{\ell}\right)$, $k \in \mathbb{N}$. 3

The problems for T become $T_k'(t) + \left(\frac{k\pi c}{\ell}\right)^2 T_k(t) = 0 \Rightarrow T_k(t) = A_k e^{-\left(\frac{k\pi c}{\ell}\right)^2 t}$,
where $A_k \in \mathbb{R}$ for $k \in \mathbb{N}$.

Combining the above and employing the superposition principle:

$$\boxed{u(x,t) = \sum_{k=1}^{\infty} X_k(x) T_k(t) = \sum_{k=1}^{\infty} C_k \sin\left(\frac{k\pi x}{\ell}\right) \exp\left(-\left(\frac{k\pi c}{\ell}\right)^2 t\right).}$$

2

b) Letting $\ell = \pi$, $c = 1$, we have that

$$u(x,0) = \sum_{k=1}^{\infty} C_k \sin(kx) = 4\sin(4x) + 5\sin(9x) \Rightarrow C_k = \begin{cases} 4 & \text{if } k=4, \\ 5 & \text{if } k=9, \\ 0 & \text{otherwise}. \end{cases}$$

Thus $\boxed{u(x,t) = 4\sin(4x)\exp(-16t) + 5\sin(9x)\exp(-81t).}$ 3

10 Q5

SECTION A
70

SECTION B

Q6. METHOD OF CHARACTERISTICS

S

Characteristic curves, parameterised by t, are defined by

$$\begin{cases} \frac{dx}{dt} = -2, & x(0) = 0 \\ \frac{dy}{dt} = y, & y(0) = s \end{cases} \implies \begin{cases} x = -2t \\ y = se^t \end{cases} \iff \begin{cases} s = ye^{\frac{x}{2}} \\ t = -\frac{x}{2}. \end{cases} \quad 4$$

Along such curves, the PDE reduces to the ODE $\underbrace{\frac{du}{dt} + u = 1}_{⊛}$, since

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} = -2u_x + yu_y. \quad 2$$

Solve ⊛ by multiplying by an integrating factor e^t, giving

$$e^t \frac{du}{dt} + e^t u = e^t \iff \frac{d}{dt}\{e^t u\} = e^t \implies e^t u = e^t + c(s) \implies u = 1 + c(s)e^{-t}$$
$$= 1 + c(ye^{x/2})e^{x/2}. \quad 4$$

The boundary condition gives $u(0,y) = 1 + c(y) = 1 + y$, so c is the identity function and thus the particular solution is $\boxed{u(x,y) = 1 + ye^x.}$ 2

12 Q6

Q7 a) PROOF OF DUHAMEL'S PRINCIPLE

B

Let $w = w(x,t,\tau)$ solve (4) and define $u(x,t) = \int_0^t w(x,t,\tau)\,d\tau$.

By the given Lemma (twice):
$$u_{tt}(x,t) = w_t(x,t,t) + \int_0^t w_{tt}(x,t,\tau)\,d\tau$$
$$= h(x,t) + \int_0^t w_{tt}(x,t,\tau)\,d\tau. \quad 3$$

Moreover, $c^2 u_{xx} = \int_0^t c^2 w_{xx}(x,t,\tau)\,d\tau = \int_0^t w_{tt}(x,t,\tau)\,d\tau$.

Therefore $u_{tt} - c^2 u_{xx} = h(x,t)$, and $u(x,0) = u_t(x,0) = 0$ for all $x \in \mathbb{R}$, and so u solves (3). 3

b) Fix $\tau > 0$. Let $s = t - \tau$ and define $U(x,s,\tau) = w(x, s+\tau, \tau)$.

Problem (4) can then be written as
$$\begin{cases} U_{ss} - c^2 U_{xx} = 0, & x \in \mathbb{R}, \ s > 0, \\ U(x,0,\tau) = 0, & x \in \mathbb{R}, \\ U_s(x,0,\tau) = h(x,\tau) & x \in \mathbb{R}. \end{cases} \quad 3$$

This has solution by d'Alembert's formula: $U(x,s,\tau) = \frac{1}{2c}\int_{x-cs}^{x+cs} h(y,\tau)\,dy$, so

$$w(x,t,\tau) = U(x,t-\tau,\tau) = \frac{1}{2c}\int_{x-c(t-\tau)}^{x+c(t-\tau)} h(y,\tau)\,dy. \quad 2$$

c) Combining parts (a) and (b), the solution to (3) is given by Duhamel's formula:

$$u(x,t) = \frac{1}{2c}\int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} h(y,\tau)\,dy\,d\tau. \quad 2$$

13 Q7

Q8. Define $\mathcal{F}\{u\} = \bar{u}(x,\xi) = \int_{-\infty}^{\infty} u(x,y)\, e^{i\xi y}\, dy$. By the Fourier derivative theorem,

FOURIER TRANSFORM FOR LAPLACE BVP

U

$\mathcal{F}\{u_{yy}\} = (-i\xi)^2\, \bar{u}(x,\xi) = -\xi^2 \bar{u}$, while $\mathcal{F}\{u_{xx}\} = (\bar{u})_{xx}$ and $\mathcal{F}\{0\} = 0$.

(have seen problems needing F.T. w.r.t. x; suspect this will be a good test of first class understanding!)

Taking Fourier transforms w.r.t. y of Laplace's equation therefore gives

$$\frac{\partial^2 \bar{u}}{\partial x^2} - \xi^2 \bar{u} = 0. \quad 4$$

This has solutions $\bar{u} = A(\xi) e^{|\xi| x} + B(\xi) e^{-|\xi| x}$. 3

For bounded solutions, $A \equiv 0$. Taking Fourier transforms w.r.t. y of the boundary condition gives $\bar{u}(0,\xi) = \int_{-\infty}^{\infty} e^{-|y| + i\xi y}\, dy = \int_{-\infty}^{0} e^{y(1+i\xi)}\, dy + \int_{0}^{\infty} e^{y(-1+i\xi)}\, dy$

$$= \frac{1}{1+i\xi}\left[e^{y(1+i\xi)}\right]_{y=-\infty}^{y=0} - \frac{1}{1-i\xi}\left[e^{y(-1+i\xi)}\right]_{y=0}^{y=\infty}$$

$$= \frac{1}{1+i\xi} + \frac{1}{1-i\xi} = \frac{2}{1+\xi^2}, \quad 3$$

implying $B(\xi) = \dfrac{2}{1+\xi^2}$. Thus $u(x,y) = \dfrac{1}{\pi}\displaystyle\int_{-\infty}^{\infty} (1+\xi^2)^{-1} e^{-|\xi| x} e^{-i\xi y}\, d\xi$. 3

13 Q8

Q9. Substituting $u(x,t) = e^{\alpha t} v(x,t)$ into $u_t - u_{xx} + 2u = e^{-2t}$ gives

HEAT EQ. CONSTANT DISSIPATION

U

$$\underline{\alpha e^{\alpha t} v} + e^{\alpha t} v_t - e^{\alpha t} v_{xx} \underline{+ 2e^{\alpha t} v} = e^{-2t}.$$

Choosing $\alpha = -2$, terms cancel, giving $v_t - v_{xx} = 1$. ⊛ 4

Seek v in the form $v = a_0 + a_1 t + b_1 x + b_2 x^2$. 4

Substitute into ⊛ : $a_1 - 2b_2 = 1$. (†)

The initial condition implies $u(x,0) = x^2 + 2x + 1 = b_2 x^2 + b_1 x + a_0$,

hence $a_0 = 1$, $b_1 = 2$, $b_2 = 1$, $a_1 = 1 + 2b_2$ (from (†)) $= 3$.

Combining the above, $u(x,t) = (1 + 3t + 2x + x^2)e^{-2t}$. 4

12 Q9

SECTION B
50